Essential Math and Intuition for Dealing with a Second-order Dynamic System – A primer and review in a nutshell for MEMS Engineers

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MEMS Central

Disclaimer: The topic of dynamics and second-order dynamic system is typically covered in a quarter in most schools – this note represents a highly condensed version. I believe it is important to cover only the most indispensable and frequently encountered concepts.

The subject of dynamic systems is difficult to learn, for the following reasons:

1. It requires good knowledge of math and good intuitive understanding of physics;
2. It involves many concepts, pretty messy math;
3. It is taught from either electrical perspective, or mechanical perspective, but rarely both.
4. Many teachers don't really understand it that well.

This review tries to cover the topic of system dynamics in a natural format – from the perspective of a learner. It is perfect for an undergraduate in engineering or a graduate student.

This manuscript reviews both the general methodology and the most common cases (under step-function acceleration, under natural response, and under sinusoidal excitation).

A unified coverage is important for student body covering a broad spectrum of majors and disciplines. Many students have learned about dynamics before. However, different areas (EE, ME) have different focuses, different terminologies, and used different notations. To make things worse, many books have made mistakes or have narrow focuses that does not discuss the big picture. We hope this note is useful.

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I. What is a dynamic system?

Most natural and man-made systems are dynamic systems – that is, their response to dynamic input (sudden force, oscillatory force) differs from that to a static (or slow changing) input. In other words, in a dynamic system, SPEED matters, the RATE of CHANGE matters.

Most people in the world have intuitive understanding about mechanical systems – springs, cars, liquid, etc. Most people, on the other hand, don't have intuitive understanding about electrical systems – from childhood we are told not to touch electricity for fear of electrocution. Hence **our starting point for discussing a dynamic system is a mechanical system**.

For example, an automobile is a dynamic system – consisting of mass, spring, and dampers. For a dynamic system, the damper providing a resistive force proportional to the velocity – this makes the behavior interesting. For a car, we like to know how much it bounces after hitting a speed bump on the road (which is related to comfort), and how soon the car stops bouncing. We also like to make sure the car does not amplify any regular bouncing oscillation from the road or a slightly imbalanced tire.

Another way to look at a dynamic system is from the energy point of view. If a system contains no energy-storing device, it is a zeroth order system. If it contains one energy-storing device, it is a first order system. If it contains two energy storing device, it is typically a second-order system. A mass-spring mechanical system is a second order system, because both the mass (potential energy) and spring (elastic energy) are energy-storage devices.

MEMS sensors are often dynamic systems. A MEMS accelerometer, for example, is a mass connected to mechanical springs. At high speed, it encounters damping when the mass collides with air molecules – this is called air damping. The sensor is often subjected to sudden forcing or oscillatory forces. Their response (e.g., magnitude of response) is a function of not only the magnitude of the forcing but also a function of the frequency. For a MEMS sensor, one would want it to stop bouncing very quickly, to have high sensitivity, and to be able to respond to high frequency oscillating inputs.

The MEMS accelerometer must be operated under critical damping conditions, to avoid excessive ringing, stemming from an impact, that may distort reading. This is certainly a non-trivial materials and manufacturing challenge. To increase the bandwidth is another goal, in order to widen the signal receivable. However, to increase dynamic range means making resonant frequency large (ω_n), which requires small mass (m) and large spring constant (K). However, these (small mass and stiffer spring) would result in reduced sensitivity.

A MEMS gyroscope (gyros) is another dynamic system. Gyros typically use Coriolis force to measure rotation rate. A mass therefore must be constantly oscillated. A rotation would introduce Coriolis-induced cross-axis motion, which is sensed with a capacitive device. The gyro is therefore under active movement. In order to save energy, it is advantageous to operate the device close to resonant frequency, so that small energy input would produce large displacement motion.

To analyze, design, and optimize a dynamic system successfully requires the following basic knowledge about math.

II. General Overview of Solutions to Dynamic Systems

For a dynamic system, the solution process typically involves four steps:

1) Write/construct the differential Governing Equation for the system;
2) Identify initial conditions (including displacement, velocity, etc) of the system. Note that sometimes both initial position and initial velocity are zero.
3) Identify proper solution methods based on the input type and system characteristics;
4) Solve the differential equation and interpret the math solutions.

Both *initial conditions* and *forcing* can move a system away from its equilibrium state (i.e., everything stays stationary). If initial conditions are applied to a system (e.g., both hands pushing down on the car bumper and then suddenly release), the response is called natural response. If forcing is applied (e.g., using both hands to rock the car with a certain frequency), the response is called forced response.

For a general case, both initial conditions and forcing can be present for the same problem. The solution to a differential equation typically consists of two parts: one part is due to the natural response to initial conditions, and another due to forced input. Both are functions of time (t). They are mathematically added together, under the *linear-system approximation*.

Another way to look at the solution is this – the solution to a system consists of a part that is transient (i.e., will gradually die down given enough time) and a part that will persist over time (called steady state response). For example, if one using both hands to push down on the bumper of a car and then suddenly releases, the car would bounce with magnitude of the bouncing gradually decreasing (transient). Eventually the car would settle at the rest position (steady state). In this case, the transient (bouncing with decreasing magnitude) and the steady state (eventually re-settling at rest position) can be understood easily. Again, the transient and steady state parts are added for a linear system.

The partition of solutions according to natural/forced or transient/steady state paradigms is very important. Initial conditions set up the natural responses – the natural responses may consist of transient and steady parts.

These are simple concepts but extremely important to keep in mind in solving future cases. A person who understands the behavior of a dynamic system understands that the transient part will eventually die down. One should not confusing the steady state part with the transient part.

III. Differential equation and cases

As promised, let's learn the basics of dynamic system analysis from a mechanical system, which may include springs, dampers, and mass.

Whether the system is a car or a MEMS moving mass, it does not matter. According to Newton's law of motion, the product of mass (m) and acceleration (a= $\ddot{x}$) is equal to ALL the net forces, including –

1. external forcing f(t), which is a function of time;
2. spring restoring force proportional to position and displacement x;
3. damping restoring force proportional to velocity $\dot{x}$.

A canonical differential equation for a second order system is

$$m\ddot{x} + C\dot{x} + Kx = f(t)$$

where the term *C* is the damping coefficient, *K* is the force constant (spring constant), and *f(t)* is the forcing function. Note this is for $t > 0$.

Besides, the system may be subject to initial conditions $x(0)$ ***and*** $\dot{x}(0)$.

This system is written in mechanical analogy. Canonic expression is obtained by dividing the equation by *m* on both side

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 = a(t)$$

(Newton's law of motion gives us $f = ma$).

If f(t)=0, the solution is called the free system solution. In this case, the solution will consists of no forced component. The system behavior consists of only natural responses.

If f(t)=Asin(ωt+θ), the system is said to be under oscillatory bias. The transient portion (due to either initial conditions or forcing) typically dies down with time and is ignored for steady-state response. In such cases, we typically are only interested in the steady state response – how the amplitude of the output is influenced by the input frequency ω.

If f(t) is an arbitraty forcing function, the solution is the response to an arbitrary forcing function and may contain both transient and steady state terms. In this case, the solution is typically obtained for the most general case.

IV. Side-bar – Basic Trigonometry, Complex Number, and Exponential Functions and Relationships

Before we start, I present two sections, one (section IV) on basic math terms, and one (section V) on basic solution methods for ordinary differential equations. For readers who are familiar with topics, these side-bar sections serve as a review.

Trigonometry functions, complex numbers, and exponential expressions are all connected, thanks to the vision of Leonhard Euler (1707-83). [Pronounced oi-ler].

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

A complex number consists of a real part *a* and an imaginary part j*b*. A complex number can be represented by the rectangular form (a+jb) or polar form (magnitude $\angle angle$).

$$j^2 = -1$$
$$a + jb = \sqrt{a^2 + b^2}\angle\theta$$

For exponential numbers, we have

$$e^{a+b} = e^a \cdot e^b$$
$$e^{a+jb} = e^a e^{jb}$$

Euler's Identity further gives

$$e^{jb} = \cos b + j \sin b$$

(The Euler's identify can be proven easily with Taylor series expansion).

V. Side-bar: Mathematics Solutions Methods

Based on the types of systems and input, there are three general solution methods

Solution Type 1: Direct Integral Solution

This is the traditional, integral way of solving a differential equation, generally taught to college freshman in Calculus. Basically, one turns the governing equation into an integral operation. One can solve the differential equation directly based on initial conditions. This method can only be used for the simplest cases, such as one without damping or without forcing.

Solution Type 2: "Trial" (Guess) Solutions

One can guess a solution form, and then plug in the trial form into the differential solution. This would allow the parameters in the trial solution to be found. Since humans have been solving differential equations for a while, we actually know what the general form of a "guess solution" is (wink wink!). So the fact that we are "guessing" should not bother you too much.

For example, for a second order system with zero damping under initial conditions but zero forcing, one may try a sinusoidal solution, $x = A\sin(2\pi ft + \theta)$. The three important parameters are magnitude *A*, frequency *f*, and phase angle θ.

For example, for a second order system with non-zero damping coefficient, the general solution would be an exponential function, Ae^{at}. The solution of a is the characteristic roots of $ma^2 + Ca + K = 0$. The solution a generally would have a negative real part, hence e^{at} is exponentially decaying overtime. Of course, the value of a may be a pure real number (the system would have exponential decay), or a pure imaginary number (the system would be sinusoidally oscillating), or a combination of real and imaginary (the system would be delaying with oscillating imposed on it).

When the input forcing term is very complex, it becomes very difficult to identify the correct trial solutions. In such cases, Solution type 3 would be invoked.

The most broad trial solution is $x(t) = C + (D_1 + D_2 t)e^{at}$. [We call this trial solution case 1, TSC1]

By setting D_2 zero, we have a slightly narrower case, still pretty applicable to most cases, is $x(t) = C + De^{at}$. [We call this trial solution case 2, TSC2]

If a is a real number, the solution is exponentially decaying;
If a is an imaginary number, by Euler's identity, the solution is oscillatory;
If a is a complex number, having both imaginary and real parts, the solution would be a combination of decaying (caused by real part) and oscillating (caused by imaginary part).

If you are not sure what trial solution you should use, use TSC1 to be safe. TSC1 is the biggest trial umbrella there.

Solution Type 3: Laplace transformation

This is the most general case. One can perform Laplace transformation on both the left and right handside of the differential equation, find the solution of the Laplace transform, and then perform inverse Laplace transformation to find the solution. This method applies to all cases, including cases that fit solution types 1 and 2. For very complex forcing and initial conditions, the process of finding reverse Laplace transform may be rather tedious and time consuming.

The Laplace solution incorporates initial conditions automatically.

Table 1: Forward Laplace Transform Table:

$x(t)$	X(s)
$\delta(t)$, unit impulse	1
$u(t)$, a step function with a magnitude of 1 (unity)	$\frac{1}{s}$
Constant c	$\frac{c}{s}$

e^{-at}	$\frac{1}{s+a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\frac{dx}{dt}$	$sX(s)-x(0)$
$\frac{d^2x}{dt^2}=\ddot{x}$	$s^2X(s)-sx(0)-\dot{x}(0)$
$ax(t)$	$aX(s)$
$e^{-at}x(t)$	$X(s+a)$

Table 2: Terminology

Notation	Name	Unit	Relations
$\ddot{x}$	Second order derivation of x, acceleration	m/s^2	
$\dot{x}$	First order derivation of x, speed	m/s	
$x(0)$	Initial position	m	
$\dot{x}(0)$	Initial speed	m/s	
m	Mass	Kg	
C	Damping coefficient	$N \cdot s/m$	$\frac{C}{m}=2\xi\omega_n$; $C=\frac{K}{\omega_r Q}$
Cr	Critical damping coefficient	$N \cdot s/m$	$Cr=2\sqrt{Km}$, $\frac{Cr}{m}=2\omega_n$
K	Force constant, spring constant	N/m	
ζ (Greek letter "Zeta")	Damping ratio, or damping factor	-	$\zeta = C/Cr=\frac{C}{2\sqrt{Km}}$
ω_n	Natural frequency (undamped), or loosely called "resonant frequency"	rad/s	$\omega_n=\sqrt{\frac{K}{m}}$
ω_d	Damped natural frequency	*Rad/s*	$\omega_d=\omega_n(\sqrt{1-2\zeta^2})$
f_n	Resonant frequency	Hz	$\omega_n=2\pi f_n$

Q	Quality factor	-	$Q = \frac{1}{2\zeta} = \frac{\sqrt{Km}}{C}$

Table 3: Four damping cases

Cases	Behavior under free vibration and initial conditions	C	ζ
Overdamped	$e^{-\omega_n t}$, exponential decay	$C > Cr$, or $C^2 > 4Km$	$\zeta > 1$
Critically damped	$te^{-\omega_n t}$	$C = Cr$, or $C^2 = 4Km$	$\xi = 1$
Under damped	$e^{-\omega_n t} \sin \omega_n t$ sinusoidal modulated delay	$C < Cr$, or $C^2 < 4Km$	$\xi < 1$
Zero damping	$\sin \omega_n t$ No decay oscillation	$C = 0$	$\xi = 0$

There are four damping cases. For most MEMS systems, the system is either under damped or critically damped.

VI. Types of solutions and types of questions about dynamic systems

There are **two ways** to look at any solutions to a dynamic system: From the INPUT perspective or from the TIME perspective.

From the input perspective: The solution of a dynamic system can be due to initial conditions or due to forcing.

- The solution due to initial conditions is called NATURAL RESPONSE.
- The solution due to forcing terms is called FORCED RESPONSE.

Now let's look at a solution from another fresh angle – the time perspective. The solution of a dynamic system can be broken into a time varying part (transient) or a time invariant part (steady state).

	Natural	Forced
Time varying, transient		
Time invariant, steady state		

The natural response consists of time varying part and steady state part.

The forced response may consist of time varying part and steady part.

The transient may be caused by natural response or forced response.
The steady state solution may be caused by natural or forced.

Some most common engineering situations for dynamic analysis are:

1. **Transient response**. A first order system (such as RC or RL circuit) undergoing a transient input. In this case, both initial conditions and forcing terms exist.
2. **Transient response.** A second order system (such as mass-spring-damper or a RLC circuit) undergoing a transient. People may be interested to know how the transient response is (e.g., how fast a car body settles after hitting a pot hole; how fast a RLC circuit settles after a switch is suddenly thrown);
3. **Steady state response.** A second order electric system (that is, basically EVERY electrically system) under sinusoidal oscillatory input, or a second order mechanical system (e.g., a rotor in a hydraulic generator) under forcing (e.g., imbalance of rotor mass). In this case, the thing people mostly interested in is how the output amplitude is correlated to the input amplitude. (In this case, people care about only the Steady State response that is only attributed to Forced input. The system is supposed to have been on for a long while, or people just don't care so much about the transient part – although sometimes the transient may cause irreversible damage to the system.)
4. **Full fledged, multiphysics simulation.** In certain cases, such as MEMS gyroscopes based on Coriolis force, the steady state and the transient responses are both of interest. Further, the electrical domain (capacitor) and the mechanical domain (displacement) also merge and must be considered in tandem.

VII. Solutions under Important Cases

Although there are many cases, depending on the damping situation, the initial conditions, and forcing types, the commonly encountered cases are not too many. Here are the cases that appear most often, either in textbooks or in real engineering situations.

Case 1: Free vibration, initial conditions $x(0)$ and $\dot{x}(0)$, no damping

The governing equation is

$$m\ddot{x} + Kx = 0$$

All three solution types would work here. The solution is

$$x(t) = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t$$

This can further be consolidated into a single sine or cosine function if you like.

The physics of the solution is that initial conditions would set off ringing (oscillation) that last forever – since there is no damping.

Case 2: Free vibration, initial conditions $x(0)$ and $\dot{x}(0)$, critical damping

The governing equation is

$$m\ddot{x} + Cr\dot{x} + Kx = 0$$

The preferred solution methods are 2 and 3. It is too complex for direct solution. The solution is

$$x(t) = x(0)e^{-\omega_n t} + (\dot{x}(0) + \omega_n x(0))te^{-\omega_n t}$$

The solution consists of decaying transients. The steady state response is zero – that is, all the effect of initial conditions eventually dies down.

The physics of this solution is that initial conditions (stored energy) sets off purely decaying solutions. The decaying term $te^{-\varpi_n t}$ actually decays faster than $e^{-\varpi_n t}$. The system has no ringing (oscillation) because it is pretty heavily damped – the initial energy gets absorbed (wasted) by the damper very rapidly.

Case 3: Free vibration, initial conditions $x(0)$ and $\dot{x}(0)$, under damped

The governing equation is the same as the previous case:

$$m\ddot{x} + C\dot{x} + Kx = 0$$

The best way to solve this is to either use solution method 2 (trial solution) or solution method 3 (Laplace method).

For the sake of beginners, we attempt to solve this by Solution methods 2. Let's assume the solution is

$$x = Ae^{at}$$

We have

$$\dot{x} = Aae^{at}$$

and

$$\ddot{x} = Aa^2 e^{at}$$

If we plug these into the governing equation, we have

$$(ma^2 + Ca + k)Ae^{at} = 0$$

Since the solution is non trivial (i.e., $Ae^{at} \neq 0$), we must have

$$ma^2 + Ca + k = 0$$

The solution to this second order equation is

$$a = \frac{-C \pm \sqrt{C^2 - 4mk}}{2m}$$

Since the system is under damped, $C^2 - 4mk < 0$, hence

$$a = \frac{-C}{2m} \pm j\frac{\sqrt{4mk - C^2}}{2m}$$

The overall solution is there

$$x = Ae^{at} = Ae^{\frac{-Ct}{2m} \pm j\frac{\sqrt{4mk-C^2}}{2m}t} = A_1 e^{\frac{-Ct}{2m}t} Ae^{+j\frac{\sqrt{4mk-C^2}}{2m}t} + A_2 e^{\frac{-C}{2m}t} Ae^{-j\frac{\sqrt{4mk-C^2}}{2m}t}$$

Now we are going to simplify this, bearing two things in mind.

(1) The solution above contains both real and imaginary parts … we will only take the real parts.

(2) $\frac{C}{2m} = \frac{\xi C_r}{2m} = \xi\omega_n \, ; \frac{\sqrt{4mk - 4\xi^2 m^2 \omega_n^2}}{2m} = \omega_n \sqrt{1-\xi^2} \equiv \omega_d$

The solution in real time is

$$x = e^{-\xi\omega_n t}\{C\cos\omega_d t + D\sin\omega_d t),$$

where the terms C and D are constants.

The solution contains both decaying terms and oscillatory terms. It is actually oscillation modulating a decaying function. Since the system is under damped, oscillatory solutions would eventually die down, unlike in Case 2, where oscillatory terms never develop, and unlike Case 1, where oscillatory terms never decay.

A typical curve of an exponential decay superimposed with an oscillation is shown below.

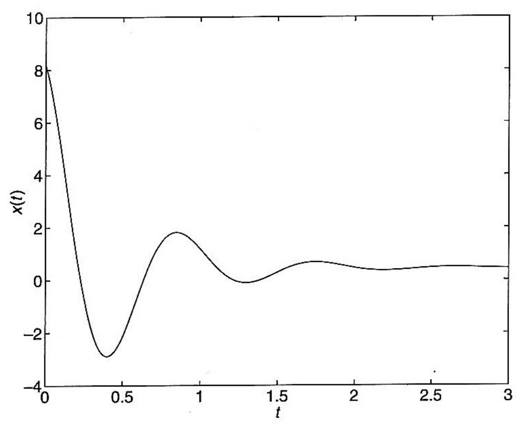


Figure 1: An exponentially decay coupled with oscillation $e^{-at}\sin(\alpha t + \theta)$**.**

Case 4: Forced response, under a step function acceleration u(t) input with magnitude *a*, zero initial conditions, critical damping

Here, an acceleration is applied at time zero and remains there. Suppose a dynamic system was initially in a weightless environment and suddenly experiences gravity. The governing equation is

$$m\ddot{x} + Cr\dot{x} + Kx = mau(t)$$

The solution is

$$x(t) = \frac{a}{\omega_n^2} - \frac{a}{\omega_n^2}e^{-\omega_n t} - \frac{a}{\omega_n^2}te^{-\omega_n t}$$

The preferred solution method is Laplace method.

The physics of this solution, is that the first term is a DC term, whereas the second and third terms are transient, exponentially decaying terms.

Especially, under steady state condition (after sufficient time),

$$x_{s.s.}(t) = \frac{a}{\omega_n^2} = \frac{a}{\frac{K}{m}} = \frac{ma}{K} = \frac{a}{\omega_n^2}.$$

The sensitivity of the accelerometer (defined as **S**) is therefore the ratio

$$S = \frac{x_{s.s}(t)}{a} = \frac{1}{\omega_n^2}$$

A system with higher resonant frequency and hence, greater bandwidth, will have a smaller response (sensitivity). The needs to have high bandwidth and high sensitivity are against one another.

VIII: Forced response under sinusoidal periodic inputs

This section can be considered an extension of the previous four cases, however, it has enough information to be isolated into a separate section. There are three unique characteristics:

(1) The forcing input is sinusoidal;
(2) WITH THESE CASES, we do not consider the transient response portion, but focus on the steady state portion.
(3) Since initial conditions can only contribute to transient, decaying solution that eventually dies down to zero (due to the damping elements in the system), we actually assume that initial conditions are ALWAYS ZERO.

Case 5: Sinusoidal input $f(t) = F\sin(\omega t) = ma\sin(\omega t)$, zero initial conditions, critical damping.

The governing equation is

$$m\ddot{x} + C_r\dot{x} + Kx = f(t) = ma\sin(\omega t)$$

The transient response (due to initial conditions and forcing) is rather complex. However, the transient response is often not the focus of analysis. In such case, we only care about the steady state part of the response. What people generally know, is that after a while, the system solution will be a sinusoidal wave with same frequency as the driving frequency. We know that when the system enters the steady state, the response MUST follow the frequency of the forcing input. Hence the steady-state solution is

$$x_{s.s.}(t) = A\sin(\omega t + \theta)$$

where the frequency will be same as driving frequency.

What people are interested is how the amplitude of the solution changes with the frequency (this is called frequency domain analysis).

If we don't care about the transient solution in general analysis, we can just focus on the steady state part. The trial solution method really does not apply anymore. The best is to use the Laplace transformation method.

Now, what I am about to write here takes a bit twist of mind, and is very important. We know we don't care about the transient behavior, and we know the initial conditions only contribute to the transient behavior. This way, we really don't care what the initial conditions are. We can set the initial conditions to zero, since it is the easiest. This makes the Laplace transformation really easy, since the Laplace transform of $\frac{dx}{dt}$ is now $sX(s)$, and the Laplace transform of $\ddot{x}$ is now $s^2 X(s)$.

We can perform Laplace transform on both sides of the governing equation, while assume all initial conditions are zero:

$$ms^2 X + C_r sX + KX = \frac{ma\omega}{s^2 + \omega^2},$$

or

$$X = \frac{ma\omega}{(s^2 + \omega^2)(ms^2 + C_r s + k)}.$$

To transform this back to the time domain is still a rather tedious operation, since the denominator is a fourth-order polynomial.

Specifically, what people are generally interested in is the ratio between the amplitude of output response (A) vs. the amplitude of input excitation (a). For this reason, we don't really even need to solve the governing equation. From now on, we will only care about the magnitude of the transfer function.

The amplification factor, or the transfer function, between the amplitude of the response A and the amplitude of the input a, is

$$T(s) = \frac{X}{a} = \frac{1}{s^2 + \frac{C_r}{m}s + \frac{k}{m}} = \frac{1}{s^2 + \omega_n^2 + 2\omega_n s}$$

Replace s with $j\omega$, we arrive at

$$T(\omega) = \frac{1}{\omega_n^2 - \omega^2 + 2j\omega_n\omega}$$

with the magnitude of $T(\omega)$ being a function of the frequency ω,

$$|T(\omega)| = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\omega^2\omega_n^2}}$$

According to this, when the frequency equals the resonant frequency, the magnitude of response is the highest (resonant peak).

Note at frequency $\omega = 0$, the magnitude of the transfer function is

$$|T(\omega)| = \frac{1}{\omega_n^2} = \frac{m}{K}$$

At resonant frequency, the magnitude is

$$T(\omega_n) = \frac{1}{2\omega_n^2} =$$

Case 6: Sinusoidal input, $f(t) = ma\sin(\omega t)$. Zero initial conditions, any damping.

The government equation is

$$m\ddot{x} + C\dot{x} + Kx = f(t) = ma(t)\,.$$

The transfer function between x and a is

$$T = \frac{X}{A} = \frac{1}{s^2 + \frac{C}{m}s + \frac{k}{m}} = \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\,.$$

If one replaces s with $j\omega$, the spectral response of T is

$$|T(\omega)| = \left|\frac{1}{\omega_n^2 - \omega^2 + j2\xi\omega\omega_n}\right| = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega^2\omega_n^2}}$$

This represents the <u>linearity ratio between output displacement and magnitude of acceleration.</u>

On way to double check the solution is this: at DC ($\omega = 0$), the magnitude of the transfer function is m/K.

The quality factor, which is inversely proportional to the damping ratio, is an indication of the amplification factor at resonance.

The magnitude of response will be linearly proportional to the DC displacement ($\frac{F}{K}$, or $\frac{mA}{K}$. The <u>linearity ratio between output displacement to DC displacement</u> is therefore

$$\frac{X}{mA/K} = \frac{\frac{X}{A}}{\frac{1}{\omega_n^2}} = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega^2\omega_n^2}}\omega_n^2 = \frac{1}{\sqrt{(1 - (\frac{\omega}{\omega_n})^2)^2 + 4\xi^2(\frac{\omega}{\omega_n})^2}}$$

No matter which way we look, at resonant conditions, the magnitude of ratio between displacement and acceleration is

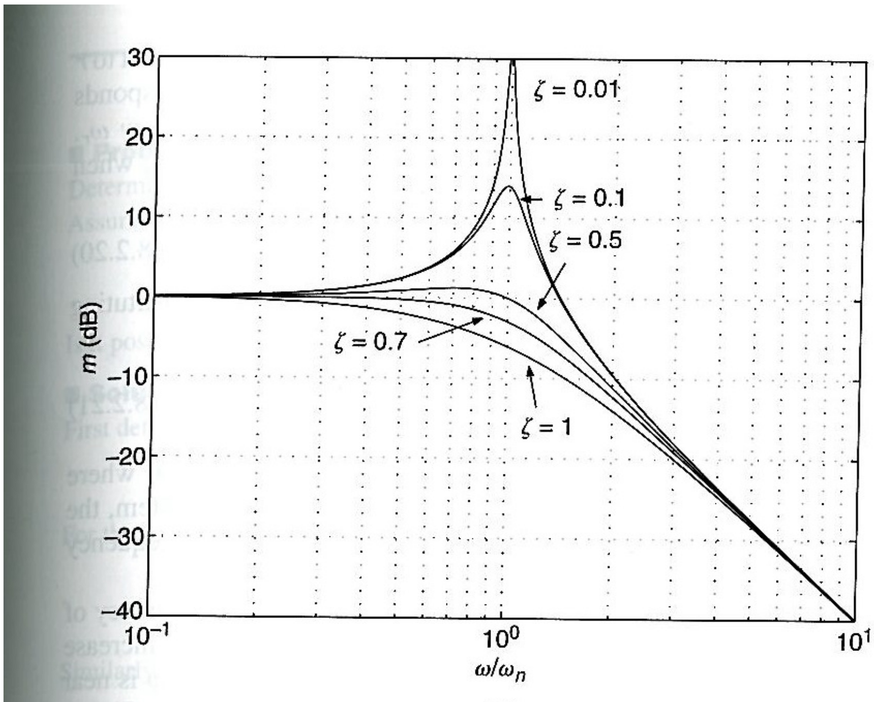
$$\frac{1}{2\xi\omega_n^2} = \frac{m}{2\xi k}\,.$$

If we compare the magnitude of output displacement to the input displacement at the resonant frequency, we have

$$\frac{x}{a/\omega_n{}^2} = \frac{1}{2\xi} = Q$$

The magnitude is not always expressed in linear scale, but using 20log scale, often referred to as Decibels, or dB.

A typical plot of a resonant behavior is shown below, with varying degree of damping:



(Fig. 8.2.4, William Palm III, System Dynamics 2nd Edition)

More In-depth discussion of second order system resonance

A second-order dynamic system has a displacement-to-static-displacement transfer function of

$$\frac{1}{\sqrt{(1-(\frac{\omega}{\omega_n})^2)^2+4\zeta^2(\frac{\omega}{\omega_n})^2}}.$$

The magnitude of displacement-acceleration transfer function is

$$\frac{1}{\sqrt{(\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2}}.$$

The maximum magnitude occurs when $\omega=\omega_d=\omega_n\sqrt{(1-2\zeta^2)}$. This term ω_d is the damped natural resonant frequency. It can be proven that the above mentioned magnitudes of transfer functions are the greatest when $\omega=\omega_d$ (for $0\le\zeta\le 0.707$). The maximum magnitude is

$$\frac{1}{\sqrt{(1-1+2\zeta^2)^2+4\zeta^2(1-2\zeta^2)}}=\frac{1}{\sqrt{4\zeta^2-4\zeta^4}}=\frac{1}{2\zeta\sqrt{1-\zeta^2}}.$$

The BANDWIDTH of a dynamic resonance system, is a very confusing term.

- If the intent of the designer is to avoid resonance, then the bandwidth is the range from 0Hz to the maximum allowable frequency where the magnitude overshoot is less than a design target;
- If the intent of the designer is to USE resonance, then the bandwidth is the width of the resonance peak.

A good discussion can be found in "System dynamics", William Palm III, second edition.

IX. Equivalency with Electrical System

Electrical systems have many similarities with mechanical systems. To build the governing equation of an electrical system, one needs to (1) find how each component behaves; and (2) use current or voltage laws to sum up these component-level behavior.

In mechanical systems, the governing law is the Newton's law: Force is the product of mass and acceleration.

In electrical systems, the governing law is the Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL). KCL says that the current running in and out of a node must sum to be zero. The KVL says that the voltage around a loop must be zero.

For a second order electrical system with R, C, and L, the general differential equation is expressed in terms of voltage or current. The basic general equations for resistors, capacitors and inductors are:

Resistor, Ohm'a law, $V = RI$.

Capacitor, $I = C_e \frac{d}{dt}V$.

Inductor, $V = L\frac{d}{dt}I$.

Here, **we use the term C_e to represent capacitance**. **We will use C_m for representing mechanical damping coefficient.** The subscript **e** is applied to avoid confusion with the symbol for damping coefficient in mechanical systems.

We will not discuss electrical system in detail here. The general equivalent circuit of a mechanical m-c-k sytem is an electrical RLC-parallel circuitry.

Scientifically and philosophically, the term voltage is the hardest to understand and explain. Over times, voltage has been called by the names of electromotive force or electrical potential. I am sure the fact that something being referred to as both a force and a potential would confuse a mechanical engineer tremendously. In mechanical engineering, force and potentials are vast distinctable concepts.

We know that there is equivalency between mechanical and electrical systems. However, this has been difficult for many.

The most easily comparable elements across the mechanical and electrical domains are the resistor in electrical domain and the damper in the electrical domain – both elements takes useful energy and dump into heat/waste. The electrical resistor and the mechanical dampers are both attenuators.

For a damper, $F_{reactive} = -C\frac{dx}{dt}$. (Note that both F and x are functions of t).

For a resistor, $Voltage = RI = R\frac{dQ}{dt}$, where Q is the overall electric charge. This suggests one way to look at **the electrical-mechanical equivalency: voltage is force, and charge is distance**). (We also cann't write it the other way, like $I = (1/R)\frac{d\,?}{dt}$ -- physically, voltage is not a derivative of anything with respect to time.)

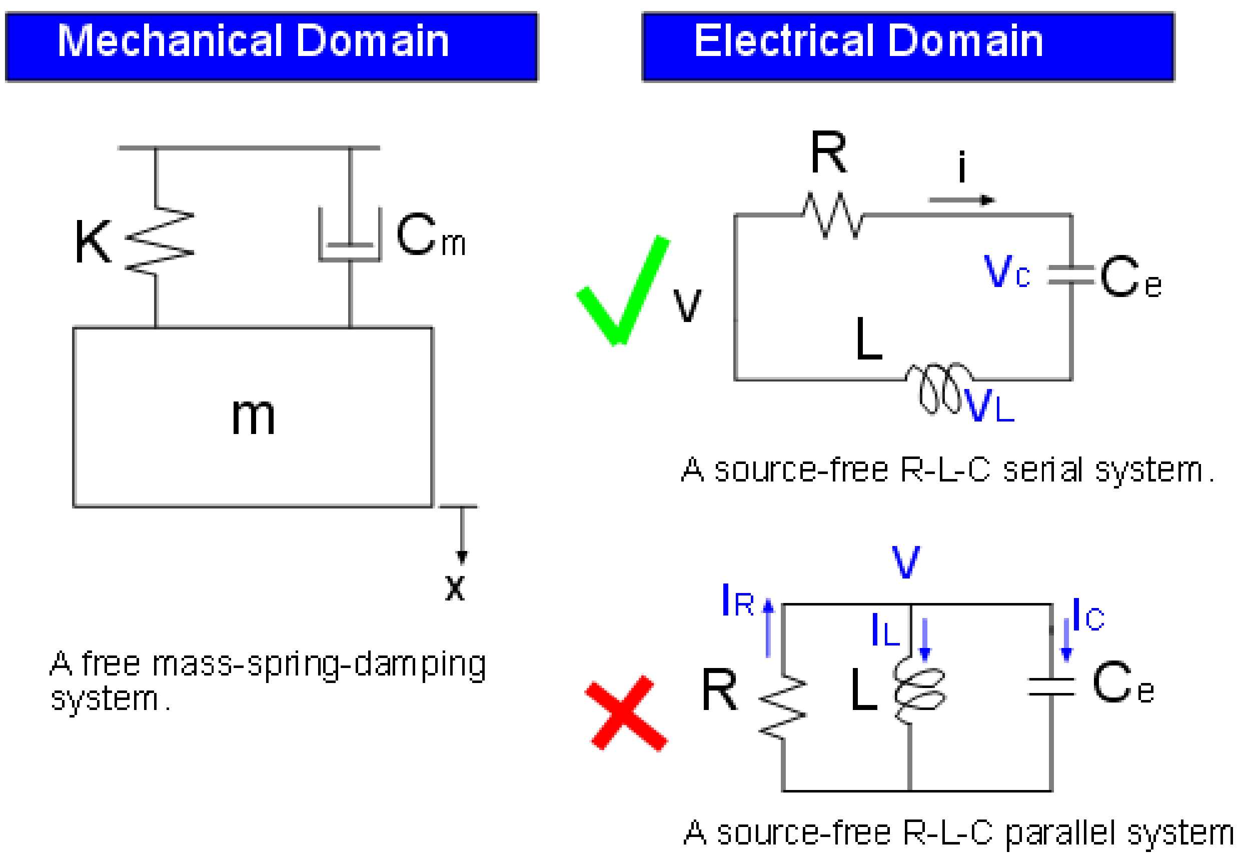


Figure 2: Mechanical and electrical system diagrams of equivalency.

The most common mechanical dynamic system is one with mass connected in parallel to both the damper and the spring (Figure 2). What is the electrical circuit equivalent? There are two circuit possibilities – RLC parallel or RLC serial. We will learn that the resistance is equivalent to damper. Considering that the mechanical system has the damping and mass having different force, the R-L-C serial arrangement is the true equivalent of the mechanical diagram to the left of Figure 2, since the voltage (i.e., force equivalent) on the resistor and other elements can be different.

However, for the sake of mathematics, we analyze both RLC serial and RLC parallel circuits here.

For the **R-L-C serial source-free configuration**, the current running through each device is the same. The resistance drops on all three devices should add up to be zero.

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int Idt - V_c(0) = 0 .$$

If one takes the derivative on both sides with respect to t, we have

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$$

In this case, the electrical resistance R is equivalent to damping coefficient C_m, the electrical inductance L is equivalent to m, and $1/C_e$ is equivalent to spring constant K.

RLC Serial Circuits electrical-mechanical equivalency		
R	L	$1/C_e$
C_m	m	K

If we use the mechanical analogy, the mechanical resonant frequency is

$$\omega_n = \sqrt{\frac{K}{m}} ,$$

and the electrical resonant frequency is

$$\omega_n = \sqrt{\frac{1/C_e}{L}} = \sqrt{\frac{1}{LC_e}} .$$

If we use the mechanical analogy, the quality factor is

$$Q = \frac{1}{2\xi} = \frac{\sqrt{Km}}{C_m} .$$

The Q for the RLC serial electrical system would be

$$Q_{electrical} = \frac{\sqrt{L/C_e}}{R} .$$

The bigger resistance, the lower the quality factor.

For a **R-L-C source-free parallel arrangement**, we know that the voltage drop on all the three elements must be identical at all times. The current running in the resistor must be the summation of the currents running in the inductor and the capacitor. The governing equation is therefore

$$\frac{V}{R}+\frac{1}{L}\int_{t}^{t_0} Vdt - I_L(0) + C\frac{dV}{dt} = 0$$

If we take derivatives on both side with respect to t, we have

$$C\frac{d^2V}{dt^2}+\frac{1}{R}\frac{dV}{dt}+\frac{1}{L}V = 0$$

In this case, the electrical conductance, which is 1/R, is equivalent to damping coefficient C_m, the electrical capacitance C_e is equivalent to m, and the inverse of electrical inductance, 1/L, is equivalent to spring constant K.

RLC Parallel Circuits electrical-mechanical equivalency		
1/R	C_e	1/L
C_m	m	K

If we use the mechanical analogy, the mechanical resonant frequency is

$$\omega_n = \sqrt{\frac{K}{m}},$$

and the electrical resonant frequency is

$$\omega_n = \sqrt{\frac{1/L}{C_e}} = \sqrt{\frac{1}{LC_e}}.$$

If we use the mechanical analogy, the quality factor is

$$Q = \frac{1}{2\xi} = \frac{\sqrt{Km}}{C_m}.$$

The Q for the electrical system would be

$$Q_{electrical} = \frac{\sqrt{C_e / L}}{1/R}.$$

(Let's check the unit to be sure. The unit of electrical quality factor is indeed unit-less, as in $[Q_{electrical}] = \dfrac{\sqrt{\left[\dfrac{Amp \cdot S}{Volt}\right]\left[\dfrac{Amp}{Volt \cdot S}\right]}}{\left[\dfrac{Amp}{Volt}\right]} = 1$)

** For a good indepth discussion of parallel resonance in circuits, refer to page 628 of "Engineering circuit analysis", Hayt, Kemmerly, and Durbin, seventh edition.
** For a good indepth discussion of series resonance in circuits, refer to page 628 of "Engineering circuit analysis", Hayt, Kemmerly, and Durbin, seventh edition.

If you have any comments or suggestions, please email Chang Liu via contact information at http://www.memscentral.com.

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